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GAUSSIAN RANDOM MEASURES

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A Gaussian random measure is a mean zero Gaussian process $\eta(A)$, indexed by sets A in a σ-field, such that $\eta(\sum A_i)=\sum \eta(A_i)$, where $\sum A_i$ indicates disjoint union and the series on the right is required to converge everywhere, so η is a random signed measure. (This is in contrast to so-called second order random measures, which only require quadratic mean convergence.) The covariance kernel of η is the signed bimeasure $\nu_0(A, B)=E\eta(A)\eta(B)$. We give a characterization of those bimeasures which are covariance kernels of Gaussian random measures, and we show that every Gaussian random measure has an exponentially integrable total variation and is a.s. absolutely continuous with respect to a fixed finite measure on the state space.

Gaussian random measure * covariance kernel * random signed measure * signed near kernel

1. Introduction

In this note we exhibit the complete structure of Gaussian random measures. Given a probability space $(\Omega, \mathcal{F}, P)$ and a measurable space $(S, \mathcal{S})$, a *Gaussian random measure* (Grm) is a Gaussian process $\eta(\omega, A)$, $A\in\mathcal{S}$, which is, for each $\omega\in\Omega$, a signed measure on $\mathcal{S}$. As is customary, we often omit ω from the notation, and, for simplicity, we assume $E\eta(A)=0$, $A\in\mathcal{S}$.

If A_i is a pairwise disjoint sequence in $\mathcal{S}$, we denote its union by $\sum A_i$. Since η is a signed measure, we have

$$\eta(\omega, \textstyle\sum A_i)=\sum \eta(\omega, A_i) \tag{1.1}$$

for all $\omega\in\Omega$. Often in the literature (e.g., [9]) one finds 'random measures' in which the series in (1.1) is only required to converge in quadratic mean (q.m.), but we are mostly concerned here with true signed measures. A mean zero Gaussian process $\eta(A)$, $A\in\mathcal{S}$, for which (1.1) converges in q.m. will be called a *Gaussian quadratic random measure* (Gqrm). The familiar example of the Brownian stochastic integral $\int_A \mathrm{d}W(t)$ shows that a Gqrm need not be a Grm. Since [3] an a.s. convergent sequence of Gaussian rvs converges in q.m., it follows that a Grm is always a Gqrm.

The *covariance kernel* ν_0 of a Grm (or Gqrm) η is defined on $\mathcal{S}\times\mathcal{S}$ by $\nu_0(A, B)=E\eta(A)\eta(B)$. (The Cartesian product $\mathcal{S}\times\mathcal{S}$ is to be distinguished from the product

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σ-field $\mathcal{S}\otimes\mathcal{S}$.) Because of q.m. convergence in (1.1), ν_0 is countably additive in each variable when the other is held fixed, i.e., ν_0 is a *signed bimeasure* on $\mathcal{S}\times\mathcal{S}$; clearly ν_0 is also finite and symmetric.

For the remainder of the paper we assume that $(S, \mathcal{S})$ is a *Lusin space*, i.e., S is measure-theoretically isomorphic to a Borel subset of a compact metric space, and $\mathcal{S}$ is the induced σ-field. This includes all the usual spaces, and allows us to use the results of [2]. In particular, the assumption implies that $\mathcal{S}$ is separable.

Here are the main results.

(1.2) Theorem. *A symmetric, finite, signed bimeasure ν_0 on $\mathcal{S}\times\mathcal{S}$ is the covariance kernel of a Grm η iff the following two conditions hold*:

(1.2.1) *ν_0 is positive definite: for any real numbers $a_1, \ldots, a_n$ and sets $A_1, \ldots, A_n \in \mathcal{S}$, $\sum_{i,j} a_i a_j \nu_0(A_i, A_j) \geq 0$,*

(1.2.2) *there is a finite constant k such that $\sum_i \nu_0^{1/2}(B_i, B_i) \leq k$ for any countable partition of S by sets $B_i \in \mathcal{S}$.*

(1.3) Theorem. *Let η be a Grm; then, a.s., $\eta(\omega, \cdot\,)$ has a finite total variation measure (denoted $|\eta|(\omega, \cdot))$ and $E\,\mathrm{e}^{\alpha|\eta|^2(S)} < \infty$ for all $\alpha < (2 \sup_{\mathcal{S}} \nu_0(A, A))^{-1}$. Moreover, there exists a Gaussian process $p(y)$, $y \in S$, having mean 0 and constant variance, such that, a.s., $\eta(A) = \int_A p(y)\mu(\mathrm{d}y)$, $A \in \mathcal{S}$, where μ is the finite, nonrandom measure $\mu(A) = E|\eta|(A)$.*

Theorems (1.2) and (1.3) give the structure of Grms. The analogue of (1.2) for Gqrms will be indicated in Section 3.

Consider now a finite measure β on $\mathcal{S}$ and let $\nu_0(A, B) = \beta(A \cap B)$. This is a finite symmetric bimeasure that satisfies (1.2.1); but, if β has no atoms, (1.2.2) cannot hold. In fact, if $0 < a < \beta(A)$, there exists a set $B \subset A$ such that $\beta(B) = a$ (Neveu [8]). Using this it is easy to construct a disjoint sequence of sets B_i such that $\sum \nu_0^{1/2}(B_i, B_i) = \infty$. This suffices to show that a Gqrm with independent increments, whose spectral measure has no atoms, cannot be a Grm. Such Gqrms arise typically in the theory of stationary processes and in empirical limit theory.

Section 2 contains a bit of background, and the proofs of (1.2) and (1.3) are given in Section 3. The motivation for this problem was a central limit theorem for certain measure-valued processes (Karr [5, Theorem (2.4)]), the conclusion of which is that the analogues of the usual normalized sample means converge in law to a 'Gaussian random measure' with mean measure 0 and a specified covariance kernel (see Karr [6, Theorem (2.9)] for comparison).

Finally we note that it is possible to require that $\eta(A)$ be Gaussian only for bounded sets $A \in \mathcal{S}$, assuming S to be a metric space; this requires using Radon measures instead of finite measures in a few places, plus some other changes. We omit the details.

2. Signed near kernels

A function K on $\Omega\times\mathcal{S}$ is a *signed near kernel* if $K(\cdot, A)$ is $\mathcal{F}$-measurable for each $A\in\mathcal{S}$, and $K(\cdot,\sum A_i)=\sum K(\cdot, A_i)$ a.s., the exceptional null set possibly depending on the disjoint sequence $A_i\in\mathcal{S}$.

Let $\mathcal{U}$ be the family of countable measurable partitions of S; the elements of $\mathcal{U}$ will be written $H=\{A_1, A_2, \ldots\}$. We write $K'(H)=\sum_i |K(A_i)|$, and ess. sup. for essential supremum (Neveu [8]).

(2.1) Lemma. (a) $E(\text{ess. sup.}_{\mathcal{U}}\, K'(H))\leq \sup_{\mathcal{U}} EK'(H)$.

(b) *If* $\text{ess. sup.}_{\mathcal{U}}\, K'(H)<\infty$ *a.s., then there exists a finite signed random measure* K_0 *on* $\Omega\times\mathcal{S}$ *such that, for each* $A\in\mathcal{S}$, $K_0(A)=K(A)$ *a.s. Moreover,* $|K_0|(S)=\text{ess. sup.}_{\mathcal{U}}\, K'(H)$ *a.s.*

For proof, see Horowitz [2, Lemma (2.14) and proof of (2.9)]. The meaning of (2.1b) is that a signed near kernel can be replaced by a 'regular version', i.e., $K_0(\sum A_i)=\sum K_0(A_i)$ with no exceptional set.

3. Proofs

Proof of (1.2). Let η be a Grm. Being Gaussian, $\eta(S)$ is finite a.s., thus $\eta(\omega,\cdot)$ is a finite signed measure a.s., and the positive, negative, and total variations η^+, η^-, $|\eta|$ are all finite a.s.

Let $\mathcal{S}_0$ be a countable field which generates $\mathcal{S}$. Then (Horowitz [2, proof of (1.1)])

$$\eta^+(\omega, A)=\sup_{B\in\mathcal{S}_0}\eta(\omega, A\cap B). \tag{3.1}$$

Since $\eta^+(S)\leq \sup_{\mathcal{S}_0}|\eta(B)|\leq|\eta|(S)<\infty$ a.s., the Landau-Shepp-Fernique theorem (see Marcus [7, II.5.5]) shows that $E\,\mathrm{e}^{\alpha(\eta^+(S))^2}<\infty$ for all $\alpha<(2\sup_{\mathcal{S}}\nu_0(A, A))^{-1}$. The same argument holds for $\eta^-(S)$, and consequently the same conclusion is valid for $|\eta|(S)$.

Since $\eta(B)$ is a normal rv, $\nu_0^{1/2}(B, B)=\sqrt{2\pi}E|\eta(B)|$. Thus, if B_i is a countable partition of S,

$$\sum_i \nu_0^{1/2}(B_i, B_i)=\sqrt{2\pi}\sum_i E|\eta(B_i)| \tag{3.2}$$

which is majorized by $\sqrt{2\pi}E|\eta|(S)\equiv k<\infty$. This proves (1.2.2). Since (1.2.1) is trivial, half of (1.2) is proven.

Let ν_0 satisfy (1.2.1)-(1.2.2). A standard argument gives the existence of a mean zero Gaussian process $\tilde{\eta}(A)$, $A\in\mathcal{S}$, such that $\nu_0(A, B)=E\tilde{\eta}(A)\tilde{\eta}(B)$. A trivial calculation shows $E(\tilde{\eta}(A+B)-\tilde{\eta}(A)-\tilde{\eta}(B))^2=0$, whence $\tilde{\eta}(A+B)=\tilde{\eta}(A)+\tilde{\eta}(B)$ a.s., i.e., $\tilde{\eta}$ is 'nearly finitely additive'.

Applying Schwarz's inequality to $\tilde{\eta}$ we have $|\nu_0(A, B)| \le \nu_0^{1/2}(A, A)\nu_0^{1/2}(B, B)$, hence, by (1.2.2),

$$\sum_{i,j} |\nu_0(A_i, B_j)| \le k^2$$

whenever $\{A_i\}$ and $\{B_j\}$ are finite partitions of S.

The result of [1] yields a finite signed measure ν on $\mathcal{S}\otimes\mathcal{S}$ such that $\nu(A\times B) = \nu_0(A, B)$ for all $A, B \in \mathcal{S}$.

Now let $B_i \in \mathcal{S}$ be a pairwise disjoint sequence. From (3.2) applied to $\tilde{\eta}$, and (1.2.2), we see that $\sum_i \tilde{\eta}(B_i)$ converges a.s. The near finite additivity of $\tilde{\eta}$ shows that $\tilde{\eta}(\sum B_i) = \sum_1^n \tilde{\eta}(B_i) + \tilde{\eta}(C_n)$ a.s., where $C_n = \sum_{n+1}^{\infty} B_i$, therefore $\tilde{\eta}(C_n)$ converges a.s. But $C_n \downarrow \emptyset$, thus $E|\tilde{\eta}(C_n)|^2 = \nu_0(C_n, C_n) = \nu(C_n \times C_n) \to 0$, hence $\tilde{\eta}(C_n) \to 0$ a.s. and $\tilde{\eta}$ is a signed near kernel. The random measure η given by Lemma (2.1) is the desired Grm.

A similar argument gives the following result for Gqrms: *a symmetric, finite, signed bimeasure ν_0 on $\mathcal{S}\times\mathcal{S}$ is the covariance kernel of a Gqrm iff* (1.2.1) *holds and $\nu_0(C_n, C_n) \to 0$ whenever $C_n \downarrow \emptyset$.*

Proof of (1.3). Only the last sentence remains to be proved. We adapt a result of Kallenberg [4] to the case of random *signed* measures.

Let K be a random signed measure on $\mathcal{S}$. We say that K *has an L^2-intensity* if the family M of finite measures μ such that $\|K(A)\|_2 \le \mu(A)$, $A \in \mathcal{S}$, is nonempty, $\|\cdot\|_2$ being the L^2-norm. (Kallenberg considered L^p-intensities, $p \ge 2$, and allowed Radon measures; similar results are possible in the present situation.)

We now show that, if $\mu, \nu \in M$, then $\mu \wedge \nu \in M$. Let $f_\mu = \mathrm{d}\mu/\mathrm{d}(\mu+\nu)$ and define f_ν similarly; then $\mathrm{d}(\mu\wedge\nu) = f_\mu \wedge f_\nu \,\mathrm{d}(\mu+\nu)$. If we now let $C = \{f_\mu \le f_\nu\}$, we have $\mu(A\cap C) = \mu\wedge\nu(A\cap C)$ and $\nu(A\cap C^c) = \mu\wedge\nu(A\cap C^c)$ for all A. Thus

$$\begin{aligned}\|K(A)\|_2 &\le \|K(A\cap C)\|_2 + \|K(A\cap C^c)\|_2\\ &\le \mu(A\cap C) + \nu(A\cap C^c) = \mu\wedge\nu(A),\end{aligned}$$

so $\mu\wedge\nu \in M$. (Kallenberg gave essentially this argument for the case of positive random measures in the preprint version of [4], but not in the published version.) The family M is thus filtering to the left, and therefore has a minimal element μ_0 [8]; μ_0 is called the L^2-intensity of K.

We then have $EK'(H) \le \mu_0(S) < \infty$, $H \in \mathcal{U}$, so that Section 2 applies. In particular, $\mu(A) = E|K|(A)$ is a finite measure on $\mathcal{S}$.

Modifying the arguments in Kallenberg [4], one can derive the following results, the proofs of which we omit:

(3.3) Theorem. *If K has an L^2-intensity μ_0, then, a.s., $K \ll \mu$, and there exists a jointly measurable version $p(\omega, y)$ of the density,*

$$K(\omega, A) = \int_A p(\omega, y)\mu(\mathrm{d}y), \quad A \in \mathcal{S}. \tag{3.3.1}$$

Moreover,

$$\int_S \|p(\cdot, y)\|_2 \mu(\mathrm{d}y) < \infty, \tag{3.3.2}$$

$$\mu_0(\mathrm{d}y) = \|p(\cdot, y)\|_2 \mu(\mathrm{d}y), \tag{3.3.3}$$

and $|K|$ *also has* L^2*-intensity* μ_0.

(3.4) Theorem. *Suppose* K *is a signed random measure on* $\mathcal{S}$ *such that* $\mu(A) = E|K|(A)$ *is finite and, a.s.,* $K \ll \mu$. *If the density* p *satisfies* (3.3.2), *then* K *has an* L^2*-intensity* μ_0 *given by* (3.3.3).

The relations $\|\eta(A)\|_2 = \sqrt{2\pi} E|\eta(A)| \leq \sqrt{2\pi} E|\eta|(A)$, where η is a Grm, show immediately that η has an L^2-intensity, so (3.3) applies to η.

By Fubini's theorem there is a set $L \in \mathcal{S}$, $\mu(L^c) = 0$, such that, for $y \in L$, $E|p(\cdot, y)| = 1$ and $p(\cdot, y) = \lim \eta(\cdot, B_n(y))/\mu(B_n(y))$ a.s., where $B_n(y) \in \mathcal{S}$ is a sequence of sets, independent of ω, which 'converges down to y' in a suitable sense (see Kallenberg [4] for the construction). Thus $p(\cdot, y)$, $y \in L$, is a mean zero Gaussian process.

Enlarging the space Ω, if necessary, let ξ be a mean zero normal rv with $E|\xi| = 1$, independent of $p(\cdot, y)$, $y \in L$. The process $p_1(\cdot, y) = p(\cdot, y)$ for $y \in L$, $= \xi$ for $y \in L^c$, is jointly measurable and satisfies the conclusion of (1.3).

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